

Seri bahan kuliah Algeo #1 - 2025

Review Matriks

Bahan kuliah IF2123 Aljabar Linier dan Geometri

**Program Studi Teknik Informatika
STEI-ITB**

Sumber:

Howard Anton & Chris Rores, *Elementary Linear Algebra*

Notasi

- Matriks berukuran $m \times n$ (m baris dan n kolom):

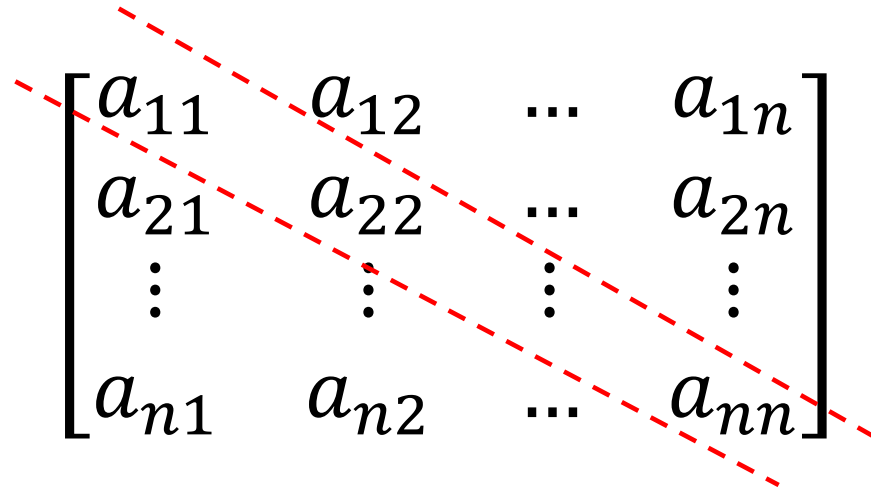
$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Jika $m = n$ maka dinamakan matriks persegi (*square matrix*) orde n

- Contoh matriks A berukuran 3×4 :

$$A = \begin{bmatrix} 3 & 2 & 4 & 6 \\ 7 & 0 & 8 & -12 \\ 13 & 11 & -1 & 0 \end{bmatrix}$$

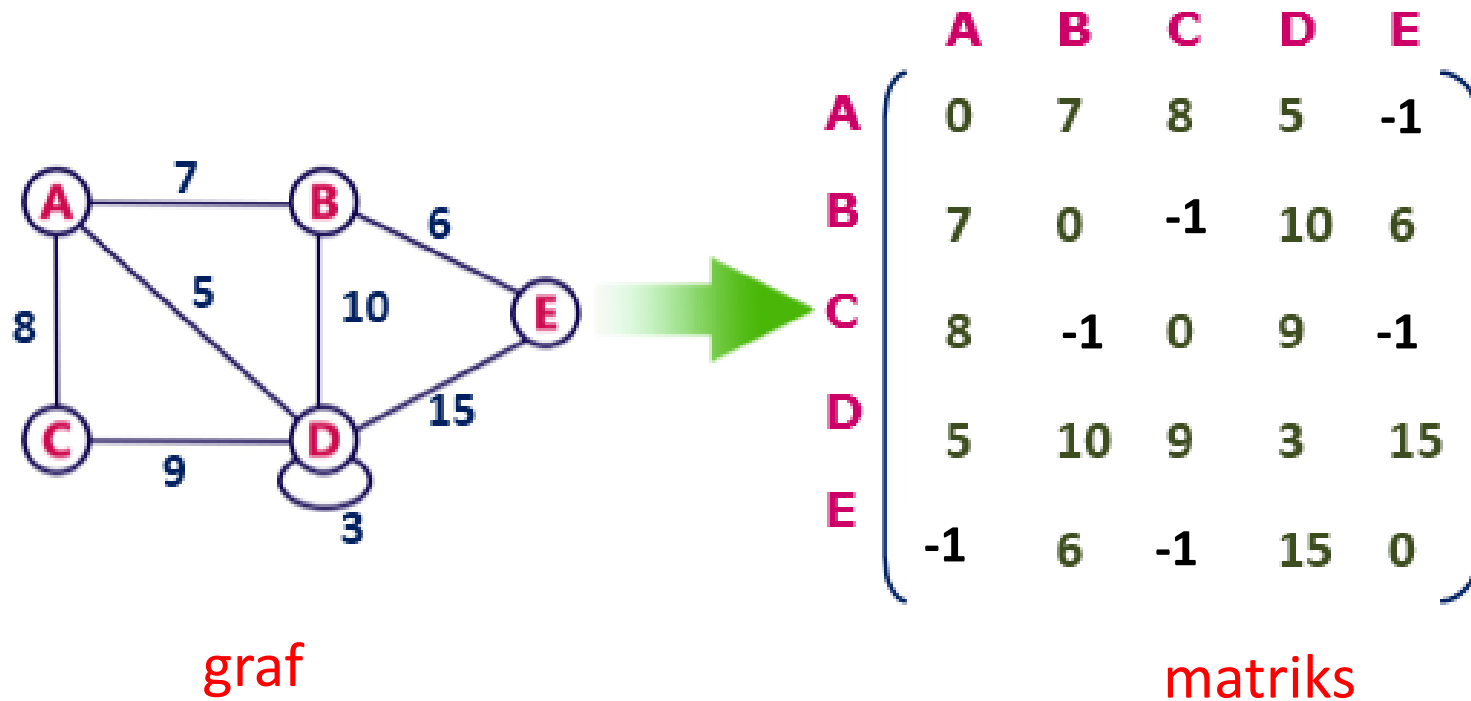
- **Diagonal utama** matriks persegi berukuran $n \times n$:


$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

- Matriks $m \times n$ tidak memiliki diagonal utama

Matriks merupakan representasi data yang sangat penting dan banyak digunakan di dalam bidang informatika, antara lain:

1. Representasi graf dengan matriks

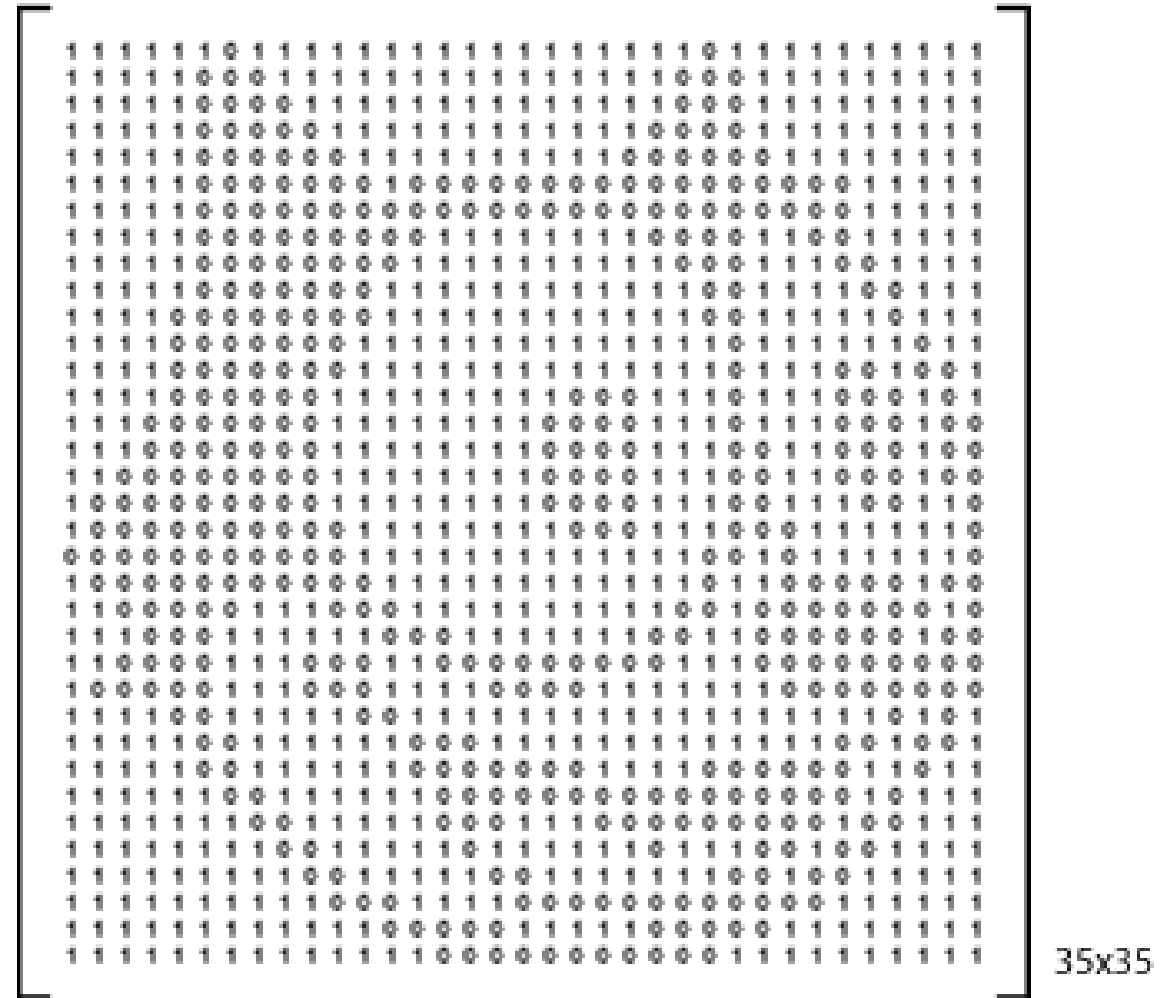


2. Representasi citra digital



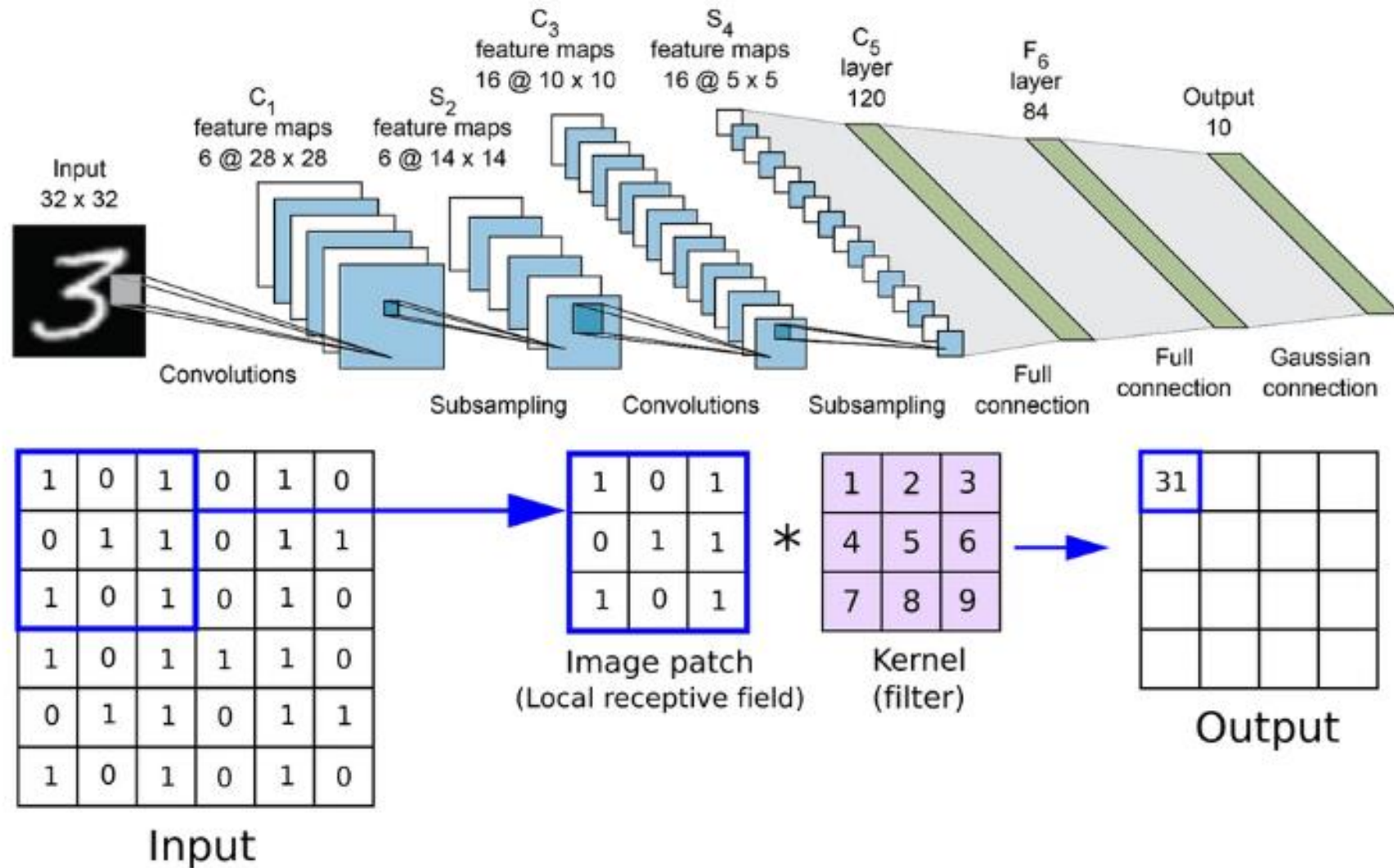
127	127	129	124	127	133	131	133	127
130	133	128	128	130	130	127	128	137
128	125	130	129	127	130	127	123	130
129	132	130	128	126	131	129	131	130
124	130	129	127	122	128	131	129	131
123	127	129	128	129	130	127	131	132

Grayscale image (nilai elemen matriks dari 0 sampai 255)



binary image (1 = putih, 0 = hitam)

3. Matriks kernel (*mask*) di dalam metode *deep learning*



4. Representasi mastriks untuk sistem persamaan linier

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_1 = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_2 = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + b_m = 0, \end{cases} \quad \longrightarrow \quad \mathbf{Ax} = \mathbf{b}$$


$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Penjumlahan Matriks

- Penjumlahan dua buah matriks $C_{m \times n} = A_{m \times n} + B_{m \times n}$

Misal $A = [a_{ij}]$

$B = [b_{ij}]$

maka $C = A + B = [c_{ij}]$, $c_{ij} = a_{ij} + b_{ij}$, $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$

- Pengurangan matriks: $C = A - B = [c_{ij}]$, $c_{ij} = a_{ij} - b_{ij}$, $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$
- Algoritma penjumlahan dua buah matriks:

```
for i ← 1 to m do
    for j ← 1 to n do
         $c_{ij} \leftarrow a_{ij} + b_{ij}$ 
    end for
end for
```

- Contoh:

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

- Maka,

$$A + B = \begin{bmatrix} -2 & 4 & 5 & 4 \\ 1 & 2 & 2 & 3 \\ 7 & 0 & 3 & 5 \end{bmatrix}, \quad A - B = \begin{bmatrix} 6 & -2 & -5 & 2 \\ -3 & -2 & 2 & 5 \\ 1 & -4 & 11 & -5 \end{bmatrix}$$

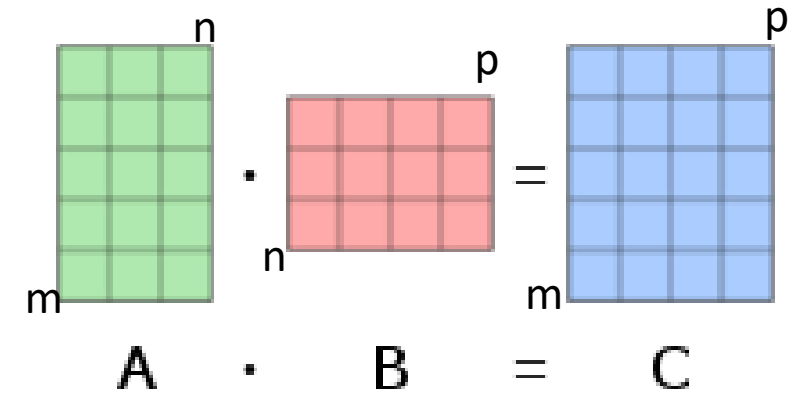
Perkalian Matriks

- Perkalian dua buah matriks $C_{m \times n} = A_{m \times n} \times B_{n \times p}$

Misal $A = [a_{ij}]$ dan $B = [b_{ij}]$

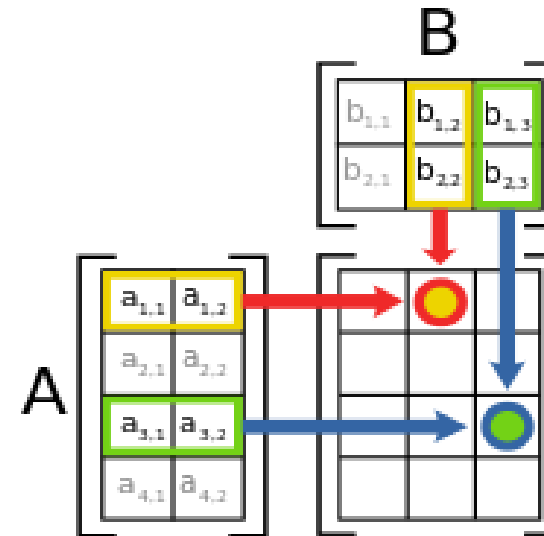
maka $C = A \times B = [c_{ij}]$, $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$

syarat: jumlah kolom A sama dengan jumlah baris B



- Algoritma perkalian dua buah matriks $C_{m \times p} = A_{m \times n} \times B_{n \times p}$

```
for i ← 1 to m do
  for j ← 1 to p do
     $c_{ij} \leftarrow 0$ 
    for k ← 1 to n do
       $c_{ij} \leftarrow c_{ij} + a_{ik} * b_{kj}$ 
    end for
  end for
end for
```



$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} a_{11}b_{11} + \cdots + a_{1n}b_{n1} & a_{11}b_{12} + \cdots + a_{1n}b_{n2} & \cdots & a_{11}b_{1p} + \cdots + a_{1n}b_{np} \\ a_{21}b_{11} + \cdots + a_{2n}b_{n1} & a_{21}b_{12} + \cdots + a_{2n}b_{n2} & \cdots & a_{21}b_{1p} + \cdots + a_{2n}b_{np} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{11} + \cdots + a_{mn}b_{n1} & a_{m1}b_{12} + \cdots + a_{mn}b_{n2} & \cdots & a_{m1}b_{1p} + \cdots + a_{mn}b_{np} \end{pmatrix}$$

- Contoh:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

maka

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

Perkalian Matriks dengan Skalar

- Misal $A = [a_{ij}]$ dan c adalah skalar
maka

$$cA = [ca_{ij}] \quad , i = 1, 2, \dots, m; j = 1, 2, \dots, n$$

- Contoh: Misalkan $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 & 7 \\ -1 & 3 & -5 \end{bmatrix}$, $C = \begin{bmatrix} 9 & -6 & 3 \\ 3 & 0 & 12 \end{bmatrix}$

maka $2A = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix}$, $(-1)B = \begin{bmatrix} 0 & -2 & -7 \\ 1 & -3 & 5 \end{bmatrix}$, $\frac{1}{3}C = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 4 \end{bmatrix}$

dan $2A - B + \frac{1}{3}C = 2A + (-1)B + \frac{1}{3}C$ ← Kombinasi linier A, B, dan C dengan koefisien 2, -1, dan 1/3

$$= \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -2 & -7 \\ 1 & -3 & 5 \end{bmatrix} + \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 2 & 2 \\ 4 & 3 & 11 \end{bmatrix}$$

Kombinasi Linier Matriks

- Perkalian matriks dapat dipandang sebagai kombinasi linier
- Misalkan:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

maka

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

- Contoh: perkalian matriks

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

dapat ditulis sebagai kombinasi linier

$$2 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

- Contoh lain: perkalian matriks

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

dapat dinyatakan sebagai kombinasi linier

$$\begin{aligned} \begin{bmatrix} 12 \\ 8 \end{bmatrix} &= 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 27 \\ -4 \end{bmatrix} &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 7 \begin{bmatrix} 4 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 30 \\ 26 \end{bmatrix} &= 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 5 \begin{bmatrix} 4 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 13 \\ 12 \end{bmatrix} &= 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 0 \end{bmatrix} \end{aligned}$$

Transpose Matriks

- Transpose matriks, $B = A^T$

$$b_{ji} = a_{ij} \quad i = 1, 2, \dots, m; j = 1, 2, \dots, n$$

- Algoritma transpose matriks:

```
for i ← 1 to m do
  for j ← 1 to n do
     $b_{ji} \leftarrow a_{ij}$ 
  end for
end for
```

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}, \quad C = [1 \ 3 \ 5], \quad D = [4]$$

$$A^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}, \quad B^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad D^T = [4]$$

- Untuk matriks persegi A berukuran $n \times n$, transpose matriks A dapat diperoleh dengan mempertukarkan elemen yang simetri dengan diagonal utama:

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 7 & 0 \\ -5 & 8 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 4 \\ 3 & 7 & 0 \\ -5 & 8 & 6 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 1 & 3 & -5 \\ -2 & 7 & 8 \\ 4 & 0 & 6 \end{bmatrix}$$

- Sifat-sifat transpose matriks

$$(a) \quad (A^T)^T = A$$

$$(b) \quad (A + B)^T = A^T + B^T$$

$$(c) \quad (A - B)^T = A^T - B^T$$

$$(d) \quad (kA)^T = kA^T$$

$$(e) \quad (AB)^T = B^T A^T$$

Trace sebuah Matriks

- Jika A adalah matriks persegi, maka *trace* matriks A adalah jumlah semua elemen pada diagonal utama, disimbolkan dengan $tr(A)$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$$tr(A) = a_{11} + a_{22} + a_{33}$$

$$tr(B) = -1 + 5 + 7 + 0 = 11$$

- Jika A bukan matriks persegi, maka $tr(A)$ tidak terdefinisi

Sifat-sifat Operasi Aritmetika Matriks

- (a) $A + B = B + A$ (Commutative law for addition)
- (b) $A + (B + C) = (A + B) + C$ (Associative law for addition)
- (c) $A(BC) = (AB)C$ (Associative law for multiplication)
- (d) $A(B + C) = AB + AC$ (Left distributive law)
- (e) $(B + C)A = BA + CA$ (Right distributive law)
- (f) $A(B - C) = AB - AC$
- (g) $(B - C)A = BA - CA$
- (h) $a(B + C) = aB + aC$
- (i) $a(B - C) = aB - aC$
- (j) $(a + b)C = aC + bC$
- (k) $(a - b)C = aC - bC$
- (l) $a(bC) = (ab)C$
- (m) $a(BC) = (aB)C = B(aC)$

Matriks Nol

- Matriks nol: matriks yang seluruh elemennya bernilai nol

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [0]$$

- Matriks nol dilambangkan dengan O
- Sifat-sifat matriks nol:

$$(a) \quad A + O = O + A = A$$

$$(b) \quad A - O = A$$

$$(c) \quad A - A = A + (-A) = O$$

$$(d) \quad OA = O$$

$$(e) \quad \text{If } cA = O, \text{ then } c = 0 \text{ or } A = O.$$

Matriks Identitas

- Matriks identitas: matriks persegi yang semua elemen bernilai 1 pada diagonal utamanya dan bernilai 0 pada posisi lainnya.

$$[1], \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Matriks identitas disimbolkan dengan I

- Perkalian matriks identitas dengan sembarang matriks menghasilkan matriks itu sendiri:

$$AI = IA = A$$

$$AI_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A$$

$$I_2A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A$$

Matriks Balikan

- Matriks balikan (*inverse*) dari sebuah matriks A adalah matriks B sedemikian sehingga

$$AB = BA = I$$

- Kita katakan A dan B merupakan balikan matriks satu sama lain

- Contoh: Misalkan $A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$

maka $AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

- Balikan matriks A disimbolkan dengan A^{-1}
- Sifat: $AA^{-1} = A^{-1}A = I$
- Untuk matriks A berukuran 2×2 , maka A^{-1} dihitung sebagai berikut:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

dengan syarat $ad - bc \neq 0$

- Nilai $ad - bc$ disebut *determinan*. Jika $ad - bc = 0$ maka matriks A tidak memiliki balikan (*not invertible*), matriks A dinamakan **matriks singular**.
- Untuk matriks berukuran $n \times n$ sembarang, balikan matriks dihitung dengan metode yang akan dibahas di dalam kuliah Algeo ini.

- Contoh:

$$A = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \longrightarrow A^{-1} = \frac{1}{7} \begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \longrightarrow \text{Tidak memiliki balikan, sebab } (-1)(-6) - (3)(2) = 0$$